

AFFINE EXTENSIONS OF PRINCIPAL ADDITIVE BUNDLES OVER A PUNCTURED SURFACE

ISAC HEDÉN

ABSTRACT. The aim of this article is to make a first step towards the classification of complex normal affine \mathbb{G}_a -threefolds X . We consider the case where the restriction of the quotient morphism $\pi: X \rightarrow S$ to $\pi^{-1}(S_*)$, where S_* denotes the complement of some regular closed point in S , is a principal \mathbb{G}_a -bundle. The variety SL_2 will be of special interest and a source of many examples. It has a natural right \mathbb{G}_a -action such that the quotient morphism $\mathrm{SL}_2 \rightarrow \mathbb{A}^2$ restricts to a principal \mathbb{G}_a -bundle over the punctured plane \mathbb{A}_*^2 .

1. INTRODUCTION

Given a complex normal affine variety X with an algebraic \mathbb{G}_a -action, the ring $\mathcal{O}(X)^{\mathbb{G}_a}$ of invariants is finitely generated if $\dim X \leq 3$ [Nag59, p. 45], so we can always define a quotient variety $X//\mathbb{G}_a := \mathrm{Spec}(\mathcal{O}(X)^{\mathbb{G}_a})$ in this case. This quotient variety is of dimension $\dim X - 1$ unless the \mathbb{G}_a -action is the trivial one. Let $\pi: X \rightarrow X//\mathbb{G}_a$ be the quotient morphism, and denote by $(X//\mathbb{G}_a)_* \subset X//\mathbb{G}_a$ the union of all open subsets $U_i \subset X//\mathbb{G}_a$ over which there is a \mathbb{G}_a -invariant trivialization $\pi^{-1}(U_i) \xrightarrow{\sim} U_i \times \mathbb{G}_a$. Then $(X//\mathbb{G}_a)_*$ is the maximal open subset $V \subset X//\mathbb{G}_a$ such that $\pi|_{\pi^{-1}(V)}: \pi^{-1}(V) \rightarrow V$ is a principal \mathbb{G}_a -bundle; this set is always nonempty.

If X is a surface, the quotient morphism is surjective and $(X//\mathbb{G}_a)_*$ is affine. In particular $\pi^{-1}((X//\mathbb{G}_a)_*) \subset X$ is equivariantly isomorphic to $(X//\mathbb{G}_a)_* \times \mathbb{G}_a$, where \mathbb{G}_a acts by translation on the second factor. It is shown in [Fie94] that complex normal affine \mathbb{G}_a -surfaces are classified by the quotient $X//\mathbb{G}_a$ and neighbourhoods of the fibers of the points in $X//\mathbb{G}_a \setminus (X//\mathbb{G}_a)_*$.

If X is a normal threefold, the quotient $S := X//\mathbb{G}_a$ is a normal affine surface, and we will study the threefolds for which $(X//\mathbb{G}_a)_* = S \setminus \{\mathbf{x}\}$ for some closed regular point $\mathbf{x} \in S$. In order to do this in a systematic way, we introduce the following notion.

Definition 1.1. Let $S_* \subset S$ be the open subvariety of an affine normal surface S which is obtained by removing a closed regular point \mathbf{x} . An affine extension of a principal \mathbb{G}_a -bundle $\pi: P \rightarrow S_*$ is a normal affine \mathbb{G}_a -variety $\hat{P} = \mathrm{Spec}(B)$ together with a morphism $\hat{\pi}: \hat{P} \rightarrow S$ and a \mathbb{G}_a -equivariant dominant open embedding $\iota: P \hookrightarrow \hat{P}$ with $\iota(P) = \hat{\pi}^{-1}(S_*)$, which makes the following diagram commute

$$\begin{array}{ccc} P & \xhookrightarrow{\iota} & \hat{P} \\ \pi \downarrow & & \downarrow \hat{\pi} \\ S_* & \xhookrightarrow{\quad} & S \end{array} \quad .$$

We will use the notation $E = \hat{\pi}^{-1}(\mathbf{x})$, $A = \mathcal{O}(P)$, $B = \mathcal{O}(\hat{P}) \xhookrightarrow{\iota^*} A$, and $\mathfrak{m}_{\mathbf{x}} \subset \mathcal{O}(S)$ for the exceptional fiber, the regular functions on P , the subalgebra of regular functions on P that extend to \hat{P} , and the maximal ideal of \mathbf{x} , respectively.

Date: April 20, 2015.

2010 Mathematics Subject Classification. 14R20.

This work was done as part of PhD studies at the Department of Mathematics, Uppsala University; the support from the Swedish graduate school in Mathematics and Computing (FMB) is gratefully acknowledged. Many thanks also go to Karl-Heinz Fieseler for his supervision.

Remark 1.2. The diagram from Definition 1.1 on the algebraic side looks as follows

$$\begin{array}{ccc} \mathcal{O}(P)^{\mathbb{G}_a} & \xleftarrow{\iota^*} & \mathcal{O}(\hat{P})^{\mathbb{G}_a} \\ \uparrow \simeq & & \uparrow \hat{\pi}^* \\ \mathcal{O}(S_*) & \xleftarrow{\simeq} & \mathcal{O}(S) \end{array}$$

if we take restrictions to the invariant algebras. It follows in particular that $\hat{P} // \mathbb{G}_a = S$ and that $\hat{\pi}: \hat{P} \rightarrow S$ is the quotient morphism.

To start with, we devote section 2 to extensions of the trivial principal \mathbb{G}_a -bundle $S_* \times \mathbb{G}_a \rightarrow S_*$.

Theorem 1. *For an affine extension \hat{P} of the trivial bundle $S_* \times \mathbb{G}_a$, the morphism ι extends to a morphism $j: S \times \mathbb{G}_a \rightarrow \hat{P}$ which is either an open embedding or contracts $\{\mathbf{x}\} \times \mathbb{G}_a$ to a singular point $p_0 \in \hat{P}$. In the first case either j is an isomorphism or $E = j(\{\mathbf{x}\} \times \mathbb{G}_a) \cup E_{(2)}$ is a disjoint union with a purely two dimensional set $E_{(2)}$; in the second case the exceptional fiber E is purely two dimensional.*

We talk accordingly of extensions of the "first kind" (j is an open embedding) and extensions of the "second kind" (j contracts $\{\mathbf{x}\} \times \mathbb{G}_a$). An obvious extension of $S_* \times \mathbb{G}_a$ of the first kind is of course $S \times \mathbb{G}_a$, but there are many others! A series of examples of smooth extensions of $\mathbb{A}_*^2 \times \mathbb{G}_a$ is presented before we move on to extensions of the second kind. Note that $S_* \times \mathbb{G}_a$ has a natural \mathbb{G}_m -action, where \mathbb{G}_m acts trivially on S_* and as $\text{Aut}(\mathbb{G}_a)$ on the fibers. We call this the vertical \mathbb{G}_m -action. Examples of affine extensions of $S_* \times \mathbb{G}_a$ such that the vertical \mathbb{G}_m -action extends to \hat{P} are also given.

Theorem 2. *Let $\hat{P} \not\cong S \times \mathbb{G}_a$ be an affine extension of $S_* \times \mathbb{G}_a$ which admits an extension of the vertical \mathbb{G}_m -action. Then \hat{P} is of the second kind and $E = \hat{P}^{\mathbb{G}_a}$ is the set of \mathbb{G}_a -fixed points. On the other hand $p_0 := j(\{\mathbf{x}\} \times \mathbb{G}_a)$ is the unique \mathbb{G}_m -fixed point in the exceptional fiber E . Furthermore all the irreducible components $E_i \hookrightarrow E$ of the exceptional fiber contain p_0 and, for each i , $E_i \setminus \{p_0\}$ is a \mathbb{G}_m -fibration over a rational curve.*

A \mathbb{G}_m -fibration in this context is an affine morphism $q: X \rightarrow Y$, where X is a \mathbb{G}_m -variety, such that the fibers are the \mathbb{G}_m -orbits and such that $q^{-1}(V) // \mathbb{G}_m \simeq V$ for each affine open subset $V \subset Y$.

The "most basic" nontrivial principal \mathbb{G}_a -bundle over \mathbb{A}_*^2 is $\text{SL}_2 \rightarrow \mathbb{A}_*^2$, $A = (a_{ij}) \mapsto (a_{11}, a_{21})$. Recall that \mathbb{G}_a embeds in SL_2 as the upper-triangular unipotent matrices; the action of \mathbb{G}_a on SL_2 is given by right multiplication. Now, if $P \rightarrow S_*$ is any nontrivial principal \mathbb{G}_a -bundle, we show in section 3 that it is possible to find a punctured neighbourhood $U_* = U \setminus \{\mathbf{x}\} \subset S$ of \mathbf{x} together with a morphism $\varphi = (g, h): U_* \rightarrow \mathbb{A}_*^2$, such that $P|_{U_*} = \varphi^*(\text{SL}_2)$. We also show that an affine extension of SL_2 always induces an extension $\tilde{P} \rightarrow U$ of $\pi^{-1}(U_*) \rightarrow U_*$, which patches together with $P \rightarrow S_*$ to an affine extension $\hat{P} \rightarrow S$. This is used as motivation for restricting our attention to extensions of SL_2 for the rest of the article.

The locally nilpotent derivation $D: B \rightarrow B$ which corresponds to the \mathbb{G}_a -action on the affine variety $\text{Spec}(B)$ can be used to define a graded algebra $\text{gr}_D(B)$, corresponding to the filtration given by $B_{\leq \nu} := \ker D^{\nu+1} \subset B$. This is done in section 4, where we associate to an affine extension $\hat{P} = \text{Spec}(B)$ its graded algebra $\text{gr}_D(B)$. Proposition 4.3 says that these graded algebras are given by a certain sequence of ideals $\{\mathfrak{m}_\nu(B)\}_{n \in \mathbb{N}}$ in $\mathcal{O}(S)$, and it turns out that they are all of the kind that appears as the algebra of an extension of the trivial bundle $S_* \times \mathbb{G}_a$ with extending vertical \mathbb{G}_m -action. We also show that $\text{gr}_D(B)$ uniquely determines B if it is generated in degree 1. Finally we formulate

Theorem 4, which could be said to be the main theorem of this work. It gives two families of graded subalgebras of $\mathrm{gr}_D(\mathcal{O}(\mathrm{SL}_2))$ that actually occur as graded algebras of affine extensions of $\mathrm{SL}_2 \rightarrow \mathbb{A}_*^2$. The construction of these two families is the topic of sections 5 and 6. The family \hat{P}_n in section 5 is indexed by a positive integer n , while the family $\hat{P}(p, q)$ in section 6 is indexed by two positive integers p, q which are relatively prime (the first construction also works for $n = 0$, but $\hat{P}_0 \simeq \hat{P}(1, 1)$ is listed in the second family instead). Both families are constructed by realizing SL_2 as a fiber bundle over some base and then enlarging the fiber – in the first case we also need to take the affinization of the obtained variety in order to get an affine extension; this corresponds to contracting a rational curve to a point. In section 5, the base is \mathbb{P}^1 and the fiber a Borel subgroup of SL_2 , whereas in section 6, the base is a Danielewski surface and the fiber \mathbb{G}_m . In the first family of extensions, the associated graded algebra is generated by its elements of degree 1, and thus uniquely determines the extension. In fact, any other SL_2 -extension is a \mathbb{G}_a -equivariant modification of one of the \hat{P}_n in the following sense.

Theorem 3. *For any affine extension \hat{P} of $P = \mathrm{SL}_2$, there exists a \mathbb{G}_a -equivariant birational morphism $\hat{P} \rightarrow \hat{P}_n$ for some $n \in \mathbb{N}$.*

In the two constructed families of affine extensions of SL_2 , the exceptional fiber consists of \mathbb{G}_a -fixed points only. In the following section 7, we obtain further extensions with a free action of \mathbb{G}_a on the exceptional fiber, as well as with a 1-dimensional fixed point set (the case of isolated fixed points for a \mathbb{G}_a -action on an affine variety is not possible).

2. EXTENSIONS OF THE TRIVIAL \mathbb{G}_a -BUNDLE

Let $P := S_* \times \mathbb{G}_a$ be the trivial \mathbb{G}_a -bundle, let $A := \mathcal{O}(S_*)[t]$ be its algebra of regular functions, and let \hat{P} be an affine extension. Using the canonical isomorphism $\mathcal{O}(S_* \times \mathbb{G}_a) \simeq \mathcal{O}(S \times \mathbb{G}_a)$, we can define a morphism $j: S \times \mathbb{G}_a \rightarrow \hat{P}$ by the condition $j^* = \iota^*: \mathcal{O}(\hat{P}) \rightarrow \mathcal{O}(S)[t]$. Note that $j: S \times \mathbb{G}_a \rightarrow \hat{P}$ extends $\iota: S_* \times \mathbb{G}_a \rightarrow \hat{P}$, and that $B := \mathcal{O}(\hat{P}) \subset A$ is a subalgebra (via ι^*).

The algebraic way of formulating the hypotheses that the \mathbb{G}_a -action on $S_* \times \mathbb{G}_a$ extends to \hat{P} and that the morphism $\hat{P} \rightarrow S$ is locally trivial over S_* , is to say that the algebra B is invariant with respect to the locally nilpotent derivation $D_t := \frac{\partial}{\partial t}: A \rightarrow A$ which corresponds to the \mathbb{G}_a -action on P , that $\mathcal{O}(S) \subset B$, and finally that $B_f = A_f$ holds for the localizations with respect to any $f \in \mathcal{O}(S)$ with $f(\mathbf{x}) = 0$.

Proof of Theorem 1. The morphism $j: S \times \mathbb{G}_a \rightarrow \hat{P}$ is equivariant since ι is, and it follows that the restriction $j|_{\{\mathbf{x}\} \times \mathbb{G}_a}$ is either injective or constant with image p_0 for some $p_0 \in \hat{P}$.

Suppose first that it is injective. Then $j: S \times \mathbb{G}_a \rightarrow \hat{P}$ is a birational morphism with finite fibers, and since \hat{P} is normal, it follows by Zariski's Main Theorem that $j: S \times \mathbb{G}_a \rightarrow \hat{P}$ is an open embedding. Furthermore, $E_2 = \hat{P} \setminus j(S \times \mathbb{G}_a)$ is purely two dimensional, being the complement of an affine open set.

If $j(\{\mathbf{x}\} \times \mathbb{G}_a) = \{p_0\}$, the point $p_0 \in \hat{P}$ is a singularity. Otherwise we could take a non-vanishing three form ω on some neighbourhood V of p_0 ; its pullback $j^*(\omega)$ would be a three form on the smooth threefold $j^{-1}(V)$, with zero set $j^{-1}(V) \cap (\{\mathbf{x}\} \times \mathbb{G}_a)$, but this is impossible for dimension reasons. For the last statement, we denote by $E_{(i)}$ the union of the i -dimensional irreducible components of E , so that $E = E_{(1)} \cup E_{(2)}$. Since \hat{P} is normal and of dimension 3, we have $\mathcal{O}(\hat{P} \setminus E_{(2)}) \simeq \mathcal{O}(\hat{P} \setminus E) = \mathcal{O}(S \times \mathbb{G}_a)$, and hence we get a factorization $\hat{P} \setminus E_{(2)} \rightarrow S \times \mathbb{G}_a \rightarrow \hat{P}$ of the inclusion $\hat{P} \setminus E_{(2)} \hookrightarrow \hat{P}$,

where the second of the maps is $j: S \times \mathbb{G}_a \rightarrow \hat{P}$. But since $j: S \times \mathbb{G}_a$ contracts $\{\mathbf{x}\} \times \mathbb{G}_a$ to a point, it follows that $E_{(1)} = \emptyset$, and $E = E_{(2)}$. \square

It follows in particular from Theorem 1 that smooth extensions of $\mathbb{A}_*^2 \times \mathbb{G}_a$ are of the first kind. An obvious example is $\hat{P} = \mathbb{A}^2 \times \mathbb{G}_a$, but there are many others, as shown by the following construction. Denote by $\underline{\mathbb{A}}^1$ the affine line with two origins, i.e. the prevariety obtained by gluing $X_1 = X_2 = \mathbb{A}^1$ along $V_1 = V_2 = \mathbb{A}_*^1$ via the identity morphism $V_1 \rightarrow V_2$, and consider a line bundle $\varphi: L \rightarrow \underline{\mathbb{A}}^1$ with trivializations $L_i \simeq X_i \times \mathbb{A}^1$ over X_i for $i = 1, 2$ and transition function $L_1 \rightarrow L_2, (x, y) \mapsto (x, x^n y)$ for some $n \in \mathbb{N}_{>0}$. Let $Q \rightarrow \underline{\mathbb{A}}^1$ be any affine nontrivial principal \mathbb{G}_a -bundle (in fact, every nontrivial \mathbb{G}_a -bundle over $\underline{\mathbb{A}}^1$ is affine [Fie94, Prop.1.4]) and let $\hat{P} := \varphi^*(Q)$ be its pullback with respect to φ ; then \hat{P} is an affine \mathbb{G}_a -variety since Q is, and the natural morphism $\hat{P} \rightarrow Q$ is affine. The principal \mathbb{G}_a -bundle $\hat{P} \rightarrow L$ has trivializations over the affine subsets $\varphi^{-1}(X_i) = L_i$, for $i = 1, 2$, and the trivial principal \mathbb{G}_a -bundle is embedded into \hat{P} via the canonical embedding $\mathbb{A}_*^2 \times \mathbb{G}_a \hookrightarrow L_2 \times \mathbb{G}_a = \mathbb{A}^2 \times \mathbb{G}_a$. The quotient morphism is $\hat{P} \rightarrow L \rightarrow \mathbb{A}^2$, where the second arrow is the identity on $L_2 \subset L$ and $(x, y) \mapsto (x, x^n y)$ on $L_1 \subset L$.

Example 2.1. We determine \hat{P} explicitly in a special case. Let $Q_i = X_i \times \mathbb{G}_a$, for $i = 1, 2$, and let Q be the principal \mathbb{G}_a -bundle which is obtained by gluing Q_1 and Q_2 along $V_1 \times \mathbb{G}_a$ and $V_2 \times \mathbb{G}_a$ via the morphism $V_1 \times \mathbb{G}_a \rightarrow V_2 \times \mathbb{G}_a, (x, t) \mapsto (x, t + \frac{1}{x})$, and let L be the line bundle given by the transition function $V_1 \times \mathbb{A}^1 \rightarrow V_2 \times \mathbb{A}^1, (x, y) \mapsto (x, xy)$. Then \hat{P} is obtained by gluing $L_1 \times \mathbb{G}_a$ and $L_2 \times \mathbb{G}_a$ along $U_1 \times \mathbb{G}_a$ and $U_2 \times \mathbb{G}_a$ via the morphism $U_1 \times \mathbb{G}_a \rightarrow U_2 \times \mathbb{G}_a, (x, y, t) \mapsto (x, xy, t + \frac{1}{x})$ with $U_i := \mathbb{A}_*^1 \times \mathbb{A}^1 \subset L_i$.

We define a morphism $\eta: \hat{P} \rightarrow \mathbb{A}^5$ by

$$\eta: (x, y, t) \mapsto \begin{cases} (x, xy, xt + 1, xyt + y, xt^2 + t) & \text{if } (x, y, t) \in L_1 \times \mathbb{G}_a \\ (x, y, xt, yt, xt^2 - t) & \text{if } (x, y, t) \in L_2 \times \mathbb{G}_a. \end{cases}$$

This is in fact a closed immersion whose image is the irreducible smooth subvariety $Z \hookrightarrow \mathbb{A}^5$ that is given by the three equations $T_1 T_4 - T_2 T_3 = T_2 T_5 + T_4 - T_3 T_4 = T_1 T_5 - T_3^2 + T_3 = 0$. The inverse morphism $\eta^{-1}: Z \rightarrow \hat{P}$ is given by

$$\eta^{-1}: (a, b, c, d, e) \mapsto \begin{cases} (a, \frac{d}{c}, \frac{e}{c}) \in L_1 \times \mathbb{G}_a & \text{if } c \neq 0 \\ (a, b, \frac{e}{c-1}) \in L_2 \times \mathbb{G}_a & \text{if } c \neq 1. \end{cases}$$

It follows that in this example we have $B = \mathbb{C}[x, y, xt, yt, xt^2 - t] \subset \mathbb{C}[x, y, t]$. Note that the vertical \mathbb{G}_m -action defined on $\mathbb{A}_*^2 \times \mathbb{G}_a \subset L_2 \times \mathbb{G}_a$ does extend to $L_2 \times \mathbb{G}_a$, but not to all of \hat{P} .

We now pass on to the second kind of extensions of the trivial \mathbb{G}_a -bundle.

Lemma 2.2. *An affine extension \hat{P} of $S_* \times \mathbb{G}_a$ is of the second kind if and only if B is a subalgebra of $\mathcal{O}(S) \oplus \bigoplus_{\nu=1}^{\infty} \mathfrak{m}_{\mathbf{x}} t^{\nu}$.*

Proof. Note that the (non finitely generated) algebra $\mathcal{O}(S) \oplus \bigoplus_{\nu=1}^{\infty} \mathfrak{m}_{\mathbf{x}} t^{\nu} \subset \mathcal{O}(S)[t]$ consists exactly of those functions on $S \times \mathbb{G}_a$ that are constant along $\{\mathbf{x}\} \times \mathbb{G}_a$. Suppose that $j(\{\mathbf{x}\} \times \mathbb{G}_a) = \{p_0\}$ for some $p_0 \in \hat{P}$. Then $j^*(f)(\{\mathbf{x}\} \times \mathbb{G}_a) = \{f(p_0)\}$ for all $f \in \mathcal{O}(\hat{P})$, so $B \subset \mathcal{O}(S) \oplus \bigoplus_{\nu=1}^{\infty} \mathfrak{m}_{\mathbf{x}} t^{\nu}$. Conversely, if $j^*(f)$ is constant along $\{\mathbf{x}\} \times \mathbb{G}_a$ for all $f \in \mathcal{O}(\hat{P})$, it follows that j contracts $\{\mathbf{x}\} \times \mathbb{G}_a$ since $\mathcal{O}(\hat{P})$ separates points in \hat{P} . \square

Lemma 2.3. *Let $\hat{P} \not\cong S \times \mathbb{G}_a$ be an affine extension of $S_* \times \mathbb{G}_a$ for which the vertical \mathbb{G}_m -action extends. Then $B = \mathcal{O}(S) \oplus \bigoplus_{\nu=1}^{\infty} \mathfrak{m}_{\nu} t^{\nu}$, with a decreasing sequence of $\mathfrak{m}_{\mathbf{x}}$ -primary ideals $\mathfrak{m}_{\nu} \subset \mathcal{O}(S)$.*

Proof. We know that B is a graded subalgebra of $\mathcal{O}(S)[t]$ with respect to the t -grading since the \mathbb{G}_m -action extends to \hat{P} , hence it is either $\mathcal{O}(S)[t]$ or has the given form. The sequence $(\mathfrak{m}_\nu)_{\nu \in \mathbb{N}_{>0}}$ is decreasing since, by assumption, B is invariant with respect to the locally nilpotent derivation $D_t: \mathcal{O}(S)[t] \rightarrow \mathcal{O}(S)[t]$ which corresponds to the \mathbb{G}_a -action on $S_* \times \mathbb{G}_a$. Finally we know that $A_f = B_f$ for all $f \in \mathcal{O}(S)$ with $f(\mathbf{x}) = 0$, and it follows that $\mathcal{O}(S)_f \oplus \bigoplus_{\nu=1}^{\infty} \mathfrak{m}_\nu \mathcal{O}(S)_f t^\nu = B_f = A_f = \mathcal{O}(S)_f \oplus \bigoplus_{\nu=1}^{\infty} \mathcal{O}(S)_f t^\nu$. In particular $\mathcal{O}(S)_f = \mathfrak{m}_\nu \mathcal{O}(S)_f$, so the support of the \mathfrak{m}_ν is contained in \mathbf{x} . \square

Corollary 2.4. *If $\hat{P} \not\simeq S \times \mathbb{G}_a$ is an affine extension of $S_* \times \mathbb{G}_a$ such that the vertical \mathbb{G}_m -action extends to \hat{P} , then \hat{P} is of the second kind.*

Proof. This is an immediate consequence of Lemma 2.2 and Lemma 2.3, since $\mathfrak{m}_\nu \subset \mathfrak{m}_\mathbf{x}$ for all $\nu \in \mathbb{N}_{>0}$, the ideals \mathfrak{m}_ν being $\mathfrak{m}_\mathbf{x}$ -primary. \square

Suppose that $B = \mathcal{O}(S) \oplus \bigoplus_{\nu=1}^{\infty} \mathfrak{m}_\nu t^\nu$ is the algebra of an affine extension \hat{P} of $S_* \times \mathbb{G}_a$, as in Lemma 2.3. Then we can form the projective spectrum $\text{Proj}(B)$; indeed, it can be thought of as the set of nontrivial \mathbb{G}_m -orbits in $\hat{P} = \text{Spec}(B)$. If B is generated in degree 1, i.e. if $B = \mathcal{O}(S) \oplus \bigoplus_{\nu=1}^{\infty} \mathfrak{b}^\nu t^\nu$ for some ideal $\mathfrak{b} \subset \mathcal{O}(S)$, the natural map $\hat{P} \setminus \hat{P}^{\mathbb{G}_m} \rightarrow \text{Proj}(B)$ is even a locally trivial \mathbb{G}_m -principal bundle and $\text{Proj}(B)$ is just the blowup of S at the ideal \mathfrak{b} .

Proof of Theorem 2. We have seen already in Corollary 2.4 that $j: S \times \mathbb{G}_a \rightarrow \hat{P}$ contracts $\{\mathbf{x}\} \times \mathbb{G}_a$. Denote $Z \rightarrow \text{Proj}(B)$ a resolution of the singularities. Then $Z \rightarrow S$ is a composition of blowups at regular points, hence the zero fiber has irreducible components isomorphic to \mathbb{P}^1 . Thus the irreducible components of the zero fiber of $\text{Proj}(B) \rightarrow S$ are dominated by \mathbb{P}^1 , hence are rational curves. It remains to show that \mathbb{G}_a acts trivially on E . The standard action of the affine group $\mathbb{G}_a \rtimes_\sigma \mathbb{G}_m$ on $\mathbb{G}_a \simeq \mathbb{A}^1$ yields an action on $P \simeq S_* \times \mathbb{G}_a$ extending to \hat{P} . Its orbits have at most dimension 1, since that holds on P and P is dense in \hat{P} . Assume that there is a nontrivial \mathbb{G}_a -orbit $\mathbb{G}_a * x \hookrightarrow E$. Then we have even

$$(\mathbb{G}_a \rtimes_\sigma \mathbb{G}_m)x = \mathbb{G}_a * x,$$

the left hand side being irreducible and one-dimensional, hence it is the union of the singular point $p_0 \in E$ and a \mathbb{G}_m -orbit. But since E is purely two dimensional, there are infinitely many such orbits – a contradiction, since different orbits are disjoint. \square

3. PULLBACKS AND EXTENSIONS OF SL_2

Principal \mathbb{G}_a -bundles over \mathbb{A}_*^2 , also studied in [DuFi11], are classified by $H^1(\mathbb{A}_*^2, \mathcal{O}_{\mathbb{A}_*^2})$. The "most basic" nontrivial among these is SL_2 , whose cocycle with respect to the open cover $\mathbb{A}_*^2 = \mathbb{A}_x^2 \cup \mathbb{A}_y^2$ is given by $(xy)^{-1} \in H^1(\mathbb{A}_*^2, \mathcal{O}_{\mathbb{A}_*^2}) \simeq x^{-1}y^{-1}\mathbb{C}[x^{-1}, y^{-1}]$. Proposition 3.1 states that every nontrivial principal \mathbb{G}_a -bundle $\pi: P \rightarrow S_*$ locally can be realized as a pullback of SL_2 with respect to a certain morphism $\varphi: U_* \rightarrow \mathbb{A}_*^2$, where U is an affine open neighbourhood of \mathbf{x} and $U_* := U \setminus \{\mathbf{x}\}$. Using this representation of $P|_{U_*} \rightarrow U_*$ as a pullback of SL_2 around $\mathbf{x} \in S$, we obtain an affine extension $\hat{\pi}: \hat{P} \rightarrow S$ for every affine extension we can find for SL_2 ; this is an direct consequence of Proposition 3.4.

Proposition 3.1. *For any nontrivial principal \mathbb{G}_a -bundle $\pi: P \rightarrow S_*$ there is a neighbourhood U of \mathbf{x} together with regular functions $g, h \in \mathcal{O}(U)$ with \mathbf{x} as their only common zero, such that*

$$P|_{U_*} \simeq \varphi^*(\text{SL}_2)$$

with the morphism $\varphi := (g, h): U_ \rightarrow \mathbb{A}_*^2$.*

Proof. We consider the subsheaf $\mathcal{F} \subset \pi_*(\mathcal{O}_P)$ on S_* of the direct image sheaf which is defined for an open affine subset $V \subset S_*$ by

$$\mathcal{F}(V) := \{f \in \mathcal{O}(\pi^{-1}(V)); D^2(f) = 0\}.$$

Here D denotes the locally nilpotent derivation which corresponds to the \mathbb{G}_a -action on $\pi^{-1}(V) \simeq V \times \mathbb{G}_a \subset P$ (see also Definition 4.1). Since \mathcal{F} is locally free of rank 2, [Hor64, Cor. 4.1.1] implies that there is a neighbourhood U of \mathbf{x} such that

$$\mathcal{F}|_{U_*} \simeq (\mathcal{O}_{U_*})^2.$$

Denote by $f_0, f_1 \in \mathcal{F}(U_*)$ the sections corresponding to $(1, 0), (0, 1) \in \mathcal{O}(U_*)^2$. When restricted to a fiber over any point in U_* , the functions f_0 and f_1 are linearly independent polynomials of degree 1, since D is the partial derivative with respect to the fiber variable, and it follows that $(0, 0)$ cannot be contained in the image of $(g, h) := (D(f_0), D(f_1)): U_* \rightarrow \mathbb{A}^2$.

Furthermore, from $D(gf_1 - hf_0) = 0$ we get that $gf_1 - hf_0$ is a nowhere vanishing function e on $\pi^{-1}(U_*)$ which is constant on the π -fibers, hence it is the pullback of a function $e \in \mathcal{O}^*(U_*)$. Replacing f_1 with $e^{-1}f_1$, we may assume that $gf_1 - hf_0 = 1$. It follows that

$$\begin{aligned} P|_{U_*} &\xrightarrow{\sim} \varphi^*(\mathrm{SL}_2) = \{(w, (u, v)) \in U_* \times \mathbb{A}^2; g(w)v - h(w)u = 1\} \\ z &\mapsto (\pi(z), f_0(z), f_1(z)) \end{aligned}$$

is an isomorphism. Note that the functions $g, h \in \mathcal{O}(U_*) = \mathcal{O}(U)$ satisfy $g(\mathbf{x}) = 0 = h(\mathbf{x})$; otherwise $P|_{U_*}$ would be trivial as well as P itself. \square

Remark 3.2. Any nontrivial principal \mathbb{G}_a -bundle $\pi: P \rightarrow \mathbb{A}_*^2$ is isomorphic to a pull-back $\varphi^*(\mathrm{SL}_2)$ with a morphism $\varphi := (g, h): \mathbb{A}_*^2 \rightarrow \mathbb{A}_*^2$. For a proof of this "global" statement for \mathbb{G}_a -bundles on \mathbb{A}_*^2 , we proceed as in the proof of Proposition 3.1, obtain that \mathcal{F} extends to a locally free sheaf $\hat{\mathcal{F}}$ on the plane and use the famous result of Quillen-Suslin [TLam06] which states that locally free $\mathcal{O}_{\mathbb{A}^2}$ -modules are free. In our situation This means $\hat{\mathcal{F}} \simeq (\mathcal{O}_{\mathbb{A}^2})^2$.

Corollary 3.3. *A principal \mathbb{G}_a -bundle over S_* is affine if and only if it is nontrivial.*

Proof. $S_* \times \mathbb{G}_a$ is not affine. If $P \rightarrow S_*$ is nontrivial and U affine, $\pi^{-1}(U)$ is affine as well. Indeed, it follows from Proposition 3.1 that

$$\pi^{-1}(U) \simeq \varphi^*(\mathrm{SL}_2) \simeq U \times_{\mathbb{A}^2} \mathrm{SL}_2$$

with respect to the morphisms $\mathrm{SL}_2 \rightarrow \mathbb{A}_*^2 \hookrightarrow \mathbb{A}^2$ and $(g, h): U \rightarrow \mathbb{A}^2$. Note that for the last isomorphism it is essential that $g(\mathbf{x}) = 0 = h(\mathbf{x})$. Thus the composite morphism $P \xrightarrow{\pi} S_* \hookrightarrow S$ is affine, hence P itself as well. \square

Our next result states that extensions of SL_2 induce extensions of $P|_{U_*}$. Using this we also get a global extensions of P by gluing.

Proposition 3.4. *Let $\hat{\varphi} = (g, h): U \rightarrow \mathbb{A}^2$ be a morphism and $\mathbf{x} \in U$ be the only common zero of $g, h \in \mathcal{O}(U)$ and $P = \varphi^*(\mathrm{SL}_2)$ with $\varphi := \hat{\varphi}|_{U_*}$. If \hat{R} is an affine extension of SL_2 , then the normalization of the reduction \hat{P} of the pull back $\hat{\varphi}^*(\hat{R}) := U \times_{\mathbb{A}^2} \hat{R}$ is an affine extension of P .*

Proof. An extension \hat{P} of P is defined by completing the pullback diagram for P into a cartesian diagram as follows. Here $\psi := \varphi^*(\pi)$ and $\hat{\psi} := \hat{\varphi}^*(\hat{\pi})$.

$$\begin{array}{ccc} P & \longrightarrow & \mathrm{SL}_2 \\ \psi \downarrow & & \downarrow \pi \\ U_* & \xrightarrow{\varphi} & \mathbb{A}_*^2 \end{array} \quad \hookrightarrow \quad \begin{array}{ccc} \hat{P} & \longrightarrow & \hat{R} \\ \hat{\psi} \downarrow & & \downarrow \hat{\pi} \\ U & \xrightarrow{\hat{\varphi}} & \mathbb{A}^2 \end{array}$$

By Lemma 3.5, the image $\hat{\varphi}(U)$ contains $0 \in \mathbb{A}^2$ as an interior point, and it follows that $P \subset \hat{P}$ is dense. \square

Lemma 3.5. *Denote by $\hat{\varphi} := (g, h): U \rightarrow \mathbb{A}^2$ the extension of the above morphism from the proof of Proposition 3.1. Then the image of $\hat{\varphi}$ contains $0 \in \mathbb{A}^2$ as an interior point.*

Proof. If $0 \in \hat{\varphi}(U)$ is not an interior point, there is an irreducible curve $C \subset \mathbb{A}^2$ through the origin, such that $C \cap \hat{\varphi}(U)$ is finite. With the embedding $\mathbb{A}^2 \hookrightarrow \mathbb{P}^2$, $(x, y) \mapsto [x : y : 1]$, $\hat{\varphi}$ induces a rational map $\bar{\varphi}: \bar{U} \rightarrow \mathbb{P}^2$, where \bar{U} is some smooth projective closure of U . By Noether-Castelnuovo's classical theorem, there is a blowup $\xi: X \rightarrow \bar{U}$ and a projective morphism $\eta: X \rightarrow \mathbb{P}^2$ such that the following diagram commutes.

$$\begin{array}{ccc}
 & X & \\
 \xi \swarrow & & \searrow \eta \\
 \bar{U} & \xrightarrow{\quad \bar{\varphi}_P \quad} & \mathbb{P}^2 \\
 \uparrow & & \uparrow \\
 U & \xrightarrow{\quad \hat{\varphi} \quad} & \mathbb{A}^2
 \end{array}$$

Note that $\xi: X \rightarrow \bar{U}$ is a finite composition of blowups at points above $\bar{U} \setminus U$, so in particular we may think of U as a subset of X . Then the inverse image $\eta^{-1}(\bar{C})$ consists of $\mathbf{x} \in U$ and finitely many further points in U and a closed subvariety of $X \setminus U$. This is a contradiction, since $\eta^{-1}(\bar{C})$ is the support of a divisor in X and thus cannot have any isolated points. \square

Remark 3.6. Unfortunately we don't know if all extensions of P can be obtained in the above way, and second, if different extensions of SL_2 induce different extensions of P .

4. THE GRADED ALGEBRA OF AN AFFINE EXTENSION

We now introduce the graded algebra, denoted $\mathrm{gr}_D(B)$ of an affine extension $\hat{\pi}: \hat{P} \rightarrow S$ of a principal \mathbb{G}_a -bundle $\pi: P \rightarrow S_*$. Motivated by Propositions 3.1 and 3.4 we will restrict our attention to affine extensions of SL_2 in the remaining sections. Hence it would be enough to develop this algebraic tool for bundles over \mathbb{A}_*^2 , but since it works completely analogously for the more general setting described in Definition 1.1, we formulate it in terms of a general punctured surface $S_* = S \setminus \{\mathbf{x}\}$ instead.

We denote by D the locally nilpotent derivation which corresponds to the structural \mathbb{G}_a -action on P .

Definition 4.1. If A is a \mathbb{C} -algebra with a locally nilpotent derivation $D: A \rightarrow A$, we define the D -filtration $(A_{\leq \nu})_{\nu \in \mathbb{N}}$ of A by $A_{\leq \nu} := \ker D^{\nu+1}$, and define the associated graded algebra $\mathrm{gr}_D(A)$ as

$$\mathrm{gr}_D(A) := \bigoplus_{\nu=0}^{\infty} A_{\leq \nu} / A_{\leq \nu-1}.$$

The "leading term" $\mathrm{gr}(f) \in \mathrm{gr}_D(A)$ of $f \in A \setminus \{0\}$ is defined as

$$\mathrm{gr}(f) := f + A_{\leq \nu-1} \in \mathrm{gr}_D(A)_{\nu},$$

where $\nu \in \mathbb{N}$ is the unique natural number such that $f \in \ker D^{\nu+1} \setminus \ker D^{\nu}$.

Remark 4.2. In our case, where A is the algebra of a principal \mathbb{G}_a -bundle over S_* , the $A_{\leq 0}$ -submodule $A_{\leq \nu} \subset A$ consists of the functions whose restriction to any fiber is a

polynomial of degree $\leq \nu$. In particular $A_{\leq 0} = \mathcal{O}(S_*) \simeq \mathcal{O}(S)$, so

$$\mathrm{gr}_D(A) \simeq \mathcal{O}(S) \oplus \bigoplus_{\nu=1}^{\infty} A_{\leq \nu} / A_{\leq \nu-1}.$$

We can always regard $\mathrm{gr}_D(A)$ as a subalgebra of the polynomial algebra $\mathcal{O}(S)[t]$ in one indeterminate t over $\mathcal{O}(S)$ as follows.

Proposition 4.3. *Let $D: A \rightarrow A$ be a locally nilpotent derivation of the \mathbb{C} -algebra A . Then the sequence of ideals \mathfrak{m}_ν , or more precisely $\mathfrak{m}_\nu(A)$, defined by*

$$\mathfrak{m}_\nu := D^\nu(A_{\leq \nu}) \hookrightarrow \mathcal{O}(S)$$

is decreasing and satisfies $\mathfrak{m}_0 = \mathcal{O}(S)$, and $\mathfrak{m}_\nu \mathfrak{m}_\mu \subset \mathfrak{m}_{\nu+\mu}$. Furthermore we have

$$\mathrm{gr}_D(A) \simeq \bigoplus_{\nu=0}^{\infty} \mathfrak{m}_\nu t^\nu \hookrightarrow \mathcal{O}(S)[t].$$

Proof. The isomorphism is induced by

$$\mathrm{gr}_D(A)_\nu \rightarrow \mathfrak{m}_\nu t^\nu, \quad a + A_{\leq \nu-1} \mapsto \frac{D^\nu a}{\nu!} t^\nu.$$

□

Example 4.4. Let $S = U$ and $A = \mathcal{O}(P)$ as in Proposition 3.1, and let $f_0, f_1 \in A_{\leq 1}$ denote functions whose restrictions generate the vector space of polynomials of degree ≤ 1 on any fiber. Then, taking $g = D(f_0)$ and $h = D(f_1)$ we have

$$A_{\leq \nu} = \bigoplus_{\alpha \in \mathbb{N}^2, |\alpha|=\nu} \mathcal{O}(S) f^\alpha$$

with the notation $f^\alpha = f_0^{\alpha_0} f_1^{\alpha_1}$ for $\alpha = (\alpha_0, \alpha_1) \in \mathbb{N}^2$ and $|\alpha| = \alpha_0 + \alpha_1$. Since $D^\nu(f^\alpha) = \alpha! g^{\alpha_0} h^{\alpha_1}$, we obtain

$$\mathrm{gr}_D(A) = \bigoplus_{\nu=0}^{\infty} \langle g, h \rangle^\nu t^\nu \subset \mathcal{O}(S)[t].$$

Let us now consider an affine extension $\hat{\pi}: \hat{P} = \mathrm{Spec}(B) \rightarrow S$ of a \mathbb{G}_a -principal bundle $P \rightarrow S_*$ with $\mathcal{O}(P) = A$. Since $D(B) \subset B$, we can form $\mathrm{gr}_D(B)$, and we get an inclusion

$$\mathrm{gr}_D(B) = \mathcal{O}(S) \oplus \bigoplus_{\nu=1}^{\infty} \mathfrak{b}_\nu t^\nu \subset \mathcal{O}(S) \oplus \bigoplus_{\nu=1}^{\infty} \mathfrak{m}_\nu t^\nu = \mathrm{gr}_D(A),$$

where $\mathfrak{b}_\nu = \mathfrak{m}_\nu(B)$.

Lemma 4.5. *The ideals $\mathfrak{m}_\nu(B)$, $\nu > 0$, of an affine extension $\hat{P} = \mathrm{Spec}(B)$ are $\mathfrak{m}_{\mathbf{x}}$ -primary, i.e. they are supported in $\{\mathbf{x}\} \subset S$, or $\mathrm{gr}_D(B) = \mathcal{O}(S)[t]$.*

Proof. First we note that if $B = \mathcal{O}(S \times \mathbb{G}_a)$ is the algebra of a trivial \mathbb{G}_a -bundle over a variety X , we have $\mathfrak{m}_\nu(B) = B^{\mathbb{G}_a} = \mathcal{O}(X)$ for all ν , i.e. the ideals $\mathfrak{m}_\nu(B)$ have empty support. It follows from the definition that the ideal sequence $\mathfrak{m}_\nu(B_f)$ of a localization at an element $f \in B^{\mathbb{G}_a}$ satisfies $\mathfrak{m}_\nu(B_f) = \mathfrak{m}_\nu(B) B_f$. Now, in our situation, $B_f = A_f$ for all $f \in \mathcal{O}(S)$ with $f(\mathbf{x}) = 0$, and A_f is indeed the algebra of a trivial \mathbb{G}_a -bundle, since $S_* \setminus V(f)$ is affine. It follows that the $\mathfrak{m}_\nu(B)$ can have no support outside $\mathbf{x} \in S$. □

Remark 4.6. If $B = \mathcal{O}(S)[f_1, \dots, f_r] \subset A$, it follows that

$$\mathcal{O}(S)[\mathrm{gr}(f_1), \dots, \mathrm{gr}(f_r)] \subset \mathrm{gr}_D(B),$$

but it is not clear that the last inclusion always is an equality; actually, it is not even clear that $\mathrm{gr}_D(B)$ has to be finitely generated.

Proposition 4.7. *The algebra $B = \mathcal{O}(\hat{P}) \subset A$ of regular functions of an affine extension \hat{P} is uniquely determined by $\text{gr}_D(B) \subset \text{gr}_D(A)$ if $\text{gr}_D(B)$ is generated by $\text{gr}_D(B)_1$ as a $\mathcal{O}(S)$ -algebra.*

Proof. If $\text{gr}_D(B)$ is finitely generated, we can take the generators to be homogeneous, i.e.

$$\text{gr}_D(B) = \mathcal{O}(S)[\text{gr}(f_1), \dots, \text{gr}(f_r)]$$

for some $f_1, \dots, f_r \in B_{\leq 1}$. Then it follows that $B = \mathcal{O}(S)[f_1, \dots, f_r]$. Indeed $B_{\leq n} \subset \mathcal{O}(S)[f_1, \dots, f_r]$ holds by induction for all $n \in \mathbb{N}$. Let $n = 1$ and $f \in A_{\leq 1}$. Then $f \in B \Leftrightarrow \text{gr}(f) \in \text{gr}_D(B)_1$ since $B_{\leq 0} = A_{\leq 0} = \mathcal{O}(S)$. This settles the case $n = 1$, and the induction step follows from the assumption on $\text{gr}_D(B)$. \square

Another consequence if the graded algebra of an affine extension $\hat{P} = \text{Spec}(B)$ of a principal \mathbb{G}_a -bundle $\pi: P \rightarrow S_*$ is generated in degree 1, concerns the \mathbb{G}_a -action on the exceptional fiber $E \subset \hat{P}$.

Proposition 4.8. *If $\hat{P} \not\simeq S \times \mathbb{G}_a$ and $\text{gr}_D(B)$ is generated in degree 1, then the exceptional fiber $E \hookrightarrow \hat{P}$ consists of fixed points only.*

Proof. Choose generators $\text{gr}(f_1), \dots, \text{gr}(f_r) \in \text{gr}_D(B)_1$. Then

$$\psi: \hat{P} \rightarrow S \times \mathbb{A}^r, z \mapsto (\pi(z), f_1(z), \dots, f_r(z))$$

is an equivariant embedding, when we endow the right hand side with the \mathbb{G}_a -action

$$\begin{aligned} (S \times \mathbb{A}^r) \times \mathbb{G}_a &\rightarrow S \times \mathbb{A}^r \\ (y, u_1, \dots, u_r, \tau) &\mapsto (y, u_1 + \tau g_1(y), \dots, u_r + \tau g_r(y)), \end{aligned}$$

where $g_i := Df_i$ regarded as function on S . Assume that $g_i(\mathbf{x}) \neq 0$ for some i . Then $\hat{P} \rightarrow S$ admits a trivialization over some neighbourhood of $\mathbf{x} \in S$. Hence it is a principal \mathbb{G}_a -bundle and thus $\hat{P} \simeq S \times \mathbb{G}_a$ since S is affine.

We remark that $\psi(\pi^{-1}(y)) \subset \{y\} \times \mathbb{A}^r$ is an affine line for $y \in S_*$ and that the exceptional fiber consists of all lines in $\{\mathbf{x}\} \times \mathbb{A}^r$, which are limits of such lines. \square

Remark 4.9. For $B \subset \tilde{B} \subset A$ we have $B = \tilde{B} \Leftrightarrow \text{gr}_D(B) = \text{gr}_D(\tilde{B})$. Namely, if $\tilde{b} \in \tilde{B}_{\leq \nu}$, there exists $b \in B_{\leq \nu}$ such that $\text{gr}(\tilde{b}) = \text{gr}(b)$, and hence $b - \tilde{b} \in \tilde{B}_{\leq \nu-1} = B_{\leq \nu-1}$. It follows by induction that $\tilde{b} \in B_{\leq \nu}$.

From now on, we specialize to $P = \text{SL}_2 \rightarrow \mathbb{A}_*^2 \subset \mathbb{A}^2$. Writing a matrix in SL_2 as $\begin{pmatrix} x & u \\ y & v \end{pmatrix}$ we may take $f_0 = u, f_1 = v$, whence $g = x, h = y$. Thus $A = \mathcal{O}(\text{SL}_2)$ satisfies

$$\text{gr}_D(A) = \bigoplus_{\nu=0}^{\infty} \langle x, y \rangle^{\nu} t^{\nu} \subset \mathbb{C}[x, y][t].$$

From Lemma 4.5 we know that the graded subalgebra $\text{gr}_D(B)$ of an affine extension $\hat{\pi}: \hat{P} \rightarrow \mathbb{A}^2$ is of the form

$$C = \mathbb{C}[x, y] \oplus \bigoplus_{\nu=1}^{\infty} \mathfrak{c}_{\nu} t^{\nu} \subset \mathbb{C}[x, y][t]$$

with ideals $\mathfrak{c}_{\nu} \subset \langle x, y \rangle^{\nu}$ for all ν .

Question 4.10. *For which decreasing sequences $(\mathfrak{c}_{\nu})_{\nu \in \mathbb{N}_{>0}}$ of $\langle x, y \rangle$ -primary ideals in $\mathbb{C}[x, y]$, is $C = \mathbb{C}[x, y] \oplus \bigoplus_{\nu=1}^{\infty} \mathfrak{c}_{\nu} t^{\nu} \subset \text{gr}_D(A)$ equal to $\text{gr}_D(B)$ for some affine extension $\hat{P} = \text{Spec}(B)$ of the principal \mathbb{G}_a -bundle $\pi: \text{SL}_2 \rightarrow \mathbb{A}_*^2$?*

Question 4.10 is partially answered by Theorem 4, whose proof is the topic of sections 5 (part 1) and 6 (part 2). It gives the answer for two families of graded subalgebras of $\text{gr}_D(\mathcal{O}(\text{SL}_2))$, which actually occur as the graded algebras of certain extensions of $\pi: \text{SL}_2 \rightarrow \mathbb{A}_*^2$.

Theorem 4. *Let $p, q \in \mathbb{N}_{>0}$, $\gcd(p, q) = 1$ and $n \in \mathbb{N}_{>0}$.*

(1) *There is a uniquely defined affine extension \hat{P}_n of SL_2 with ideals given by*

$$\mathfrak{c}_\nu = \mathfrak{m}_\nu(\hat{P}_n) = \langle x, y \rangle^{(n+2)\nu}.$$

(2) *There is an affine extension $\hat{P}(p, q)$ of SL_2 which satisfies*

$$\mathfrak{c}_\nu = \mathfrak{m}_\nu(\hat{P}(p, q)) = \bigoplus_{p\alpha+q\beta \geq (p+q)\nu} \mathbb{C}x^\alpha y^\beta.$$

Remark 4.11. (1) Note that we may extend the first family by taking $\hat{P}_0 := \hat{P}(1, 1)$.

(2) The graded subalgebra $\mathbb{C}[x, y] \oplus \bigoplus_{\nu=1}^{\infty} \langle x, y \rangle^{(n+2)\nu} t^\nu \subset \text{gr}_D(\mathcal{O}(\text{SL}_2))$ given by the ideal sequence $\mathfrak{m}_\nu(\hat{P}_n)$ is generated in degree 1, and hence uniquely determines the extension \hat{P}_n by Proposition 4.7.

(3) In the second family, we see that $x^3 \in \mathfrak{m}_2(\hat{P}(2, 1))$ but $x^3 \notin \mathfrak{m}_1(\hat{P}(2, 1))^2$. It follows that the algebra given by the above ideal sequence $\mathfrak{m}_\nu(\hat{P}(2, 1))$ is not generated by its elements in degree 1.

5. THE FIRST FAMILY OF SL_2 -EXTENSIONS

We prove part (1) of Theorem 4 by constructing a family \hat{P}_n , indexed by $n > 0$, of affine extensions of SL_2 with $\mathfrak{m}_\nu(\hat{P}_n) = \langle x, y \rangle^{(n+2)\nu}$ and in the end of the section we give the proof of Theorem 3. In order to simplify notation, we fix the positive integer $n \in \mathbb{N}_{>0}$, and tacitly understand that most of the constructions in this section depend on n . For instance we write \hat{P} rather than \hat{P}_n although \hat{P} does depend on n .

Let $B_2 = \mathbb{G}_a \rtimes \mathbb{G}_m$ denote the semidirect product with group multiplication $(a, b) \cdot (c, d) = (a + b^2c, bd)$. We also denote this product by $\rho_{(c,d)}(a, b) = \lambda_{(a,b)}(c, d)$ (i.e. ρ for right, λ for left). Note that $B_2 \simeq \mathbb{A}^1 \times \mathbb{A}_*^1$ as a variety, and that it can be realized as a (Borel) subgroup of SL_2 via

$$B_2 \hookrightarrow \text{SL}_2, \quad (\alpha, \beta) \mapsto \begin{pmatrix} \beta & \beta^{-1}\alpha \\ 0 & \beta^{-1} \end{pmatrix}.$$

Let $U_0 = \{[x : y] \in \mathbb{P}^1, x \neq 0\} \simeq \mathbb{A}^1$, and $U_1 = \{[x : y] \in \mathbb{P}^1, y \neq 0\} \simeq \mathbb{A}^1$. As usual, we take x/y and x/y as coordinates on U_0 and U_1 respectively.

Remark 5.1. The map $\text{SL}_2 \rightarrow \mathbb{P}^1$, $A = (a_{ij}) \mapsto [a_{11} : a_{21}]$ realizes SL_2 as a B_2 -principal bundle over \mathbb{P}^1 , with B_2 -equivariant trivializations given by

$$\begin{array}{l|l} \tau_0: U_0 \times B_2 \xrightarrow{\sim} (\text{SL}_2)_x & \tau_1: U_1 \times B_2 \xrightarrow{\sim} (\text{SL}_2)_y \\ (z, (u, v)) \mapsto \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \cdot \begin{pmatrix} v & v^{-1}u \\ 0 & v^{-1} \end{pmatrix} & (z, (u, v)) \mapsto \begin{pmatrix} z & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} v & v^{-1}u \\ 0 & v^{-1} \end{pmatrix} \end{array}$$

and transition function given by $\tau_1^{-1}\tau_0: (z, (u, v)) \mapsto (z^{-1}, (z, z) \cdot (u, v))$.

First we want to endow \mathbb{A}^2 with two commuting B_2 -actions (depending on n), one from the left, the other from the right, i.e. $(b_1x)b_2 = b_1(xb_2)$ for all $b_1, b_2 \in B_2$. The next step will be to find simultaneous left- and right B_2 -embeddings $B_2 \hookrightarrow \mathbb{A}^2$ with respect to the two B_2 -actions on \mathbb{A}^2 . As a first step, we consider the automorphism

$$\varphi: \mathbb{A}^1 \times \mathbb{A}_*^1 \rightarrow \mathbb{A}^1 \times \mathbb{A}_*^1, \quad (x, y) \mapsto (xy^n, y).$$

Since $B_2 = \mathbb{A}^1 \times \mathbb{A}_*^1$ as a variety, we may conjugate the group multiplication with φ and obtain B_2 -actions on $\mathbb{A}^1 \times \mathbb{A}_*^1$ as follows.

Definition 5.2. We define the B_2 -actions $*_L$ and $*_R$ by

$$\begin{array}{l} *_L: B_2 \times (\mathbb{A}^1 \times \mathbb{A}_*^1) \rightarrow \mathbb{A}^1 \times \mathbb{A}_*^1 \\ ((s, t), (x, y)) \mapsto (s, t) *_L (x, y) := \\ (\varphi \lambda_{(s, t)} \varphi^{-1})(x, y) \end{array} \quad \left| \quad \begin{array}{l} *_R: (\mathbb{A}^1 \times \mathbb{A}_*^1) \times B_2 \rightarrow \mathbb{A}^1 \times \mathbb{A}_*^1 \\ ((x, y), (s, t)) \mapsto (x, y) *_R (s, t) := \\ (\varphi \rho_{(s, t)} \varphi^{-1})(x, y) \end{array} \right.$$

Proposition 5.3. *The actions $*_L$ and $*_R$ admit extensions to a left action $\hat{*}_L$ and a right action $\hat{*}_R$ on $\mathbb{A}^2 \supset \mathbb{A}^1 \times \mathbb{A}_*^1$.*

Proof. It follows from Definition 5.2 that $*_L$ and $*_R$ are given as follows for $(s, t) \in B_2$ and $(x, y) \in \mathbb{A}^1 \times \mathbb{A}_*^1$

$$(s, t) *_L (x, y) = (st^n y^n + t^{n+2} x, ty) \quad \left| \quad (x, y) *_R (s, t) = (t^n(x + y^{n+2} s), ty),\right.$$

and these are obviously defined even for $y = 0$. \square

We will use the notation $*_L$ and $*_R$ even for the extended actions.

Remark 5.4. The morphism $\varphi: B_2 \hookrightarrow \mathbb{A}^1 \times \mathbb{A}_*^1 \subset \mathbb{A}^2$ realizes \mathbb{A}^2 both as a left- and a right B_2 -embedding with respect to the B_2 -actions on \mathbb{A}^2 given by $*_L$ and $*_R$. We shall treat that map as an inclusion and write $B_2 \subset \mathbb{A}^2$.

Now we use this B_2 -embedding in order to define a fiber bundle $Q \rightarrow \mathbb{P}^1$ associated to the fiber bundle in Remark 5.1 as follows.

Definition 5.5. We define

$$Q := \mathrm{SL}_2 \times^{B_2} \mathbb{A}^2,$$

where \mathbb{A}^2 is endowed with the left- and right B_2 -actions $*_L$ and $*_R$.

This means that as a set, Q is the orbit space with respect to the action

$$B_2 \times (\mathrm{SL}_2 \times \mathbb{A}^2) \rightarrow \mathrm{SL}_2 \times \mathbb{A}^2, \quad (b, (x, y)) \mapsto (xb^{-1}, by),$$

while it is obtained as a variety by taking the locally trivial fiber bundle from Remark 5.1 and replacing the general fiber B_2 by \mathbb{A}^2 .

Proposition 5.6. *The action of SL_2 by left multiplication on itself induces an SL_2 -action on Q . The natural inclusion $\mathrm{SL}_2 \subset Q$ coming from B_2 is thus an SL_2 -embedding.*

In order to prove Proposition 5.6, and also in order to be able to compute $\mathcal{O}(Q)$, we present the explicit description of Q in terms of gluing data:

Let $U_i \subset \mathbb{P}^1$ be as above for $i = 0, 1$, let $Q_0 = U_0 \times \mathbb{A}^2$, $Q_1 = U_1 \times \mathbb{A}^2$, and finally let $V_i = (U_0 \cap U_1) \times \mathbb{A}^2 \subset U_i \times \mathbb{A}^2$. Then Q is the variety obtained by gluing Q_0 and Q_1 along V_0 and V_1 via the morphism

$$\begin{aligned} V_0 &\rightarrow V_1 \\ (z, (u, v)) &\mapsto (z^{-1}, (z, z) *_L (u, v)) \\ &= (z^{-1}, (z^{n+1} v^n + z^{n+2} u, zv)). \end{aligned}$$

The inverse morphism is given by

$$\begin{aligned} V_1 &\rightarrow V_0 \\ (z, (u, v)) &\mapsto (z^{-1}, (-z, z) *_L (u, v)) \\ &= (z^{-1}, (-z^{n+1} v^n + z^{n+2} u, zv)). \end{aligned}$$

Proof of Proposition 5.6. The claim follows from the following formulas, which show that the SL_2 -action is algebraic. The matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2$ acts on $(z, (u, v)) \in Q_0$ as

$$A(z, (u, v)) := \begin{cases} \left(\frac{c+dz}{a+bz}, (b(a+bz), a+bz) *_L (u, v) \right) \in Q_0 & \text{if } a+bz \neq 0 \\ \left(\frac{a+bz}{c+dz}, (d(c+dz), c+dz) *_L (u, v) \right) \in Q_1 & \text{if } c+dz \neq 0, \end{cases}$$

and on $(z, (u, v)) \in Q_1$ as

$$A(z, (u, v)) := \begin{cases} \left(\frac{cz+d}{az+b}, (-a(az+b), az+b) *_L (u, v) \right) \in Q_0 & \text{if } az+b \neq 0 \\ \left(\frac{az+b}{cz+d}, (-c(cz+d), cz+d) *_L (u, v) \right) \in Q_1 & \text{if } cz+d \neq 0. \end{cases}$$

□

Remark 5.7. The right B_2 -action $*_R$ has the Q_i as invariant subsets; it is given on Q_i for $i = 0, 1$ by

$$(z, (u, v)) *_R (s, t) = (z, (u, v) *_R (s, t)),$$

and it is well defined because of the fact that the left- and the right action commute. The \mathbb{G}_a -action on Q induced by $*_R$ via the inclusion $\mathbb{G}_a \simeq \mathbb{G}_a \times \{1\} \subset B_2$ is given on Q_i by $(u, v) *_R (s, 1) = (u + v^{n+2}s, v)$, and this action extends the structural \mathbb{G}_a -bundle action on $\text{SL}_2 \subset Q$.

We now take the affinization of Q , i.e. we take $\text{Spec}(\mathcal{O}(Q))$. This construction is described in detail, and it turns out that it is given by the contraction of a curve $C \subset Q$ which is isomorphic to \mathbb{P}^1 to a point. Indeed the right B_2 -action on Q restricts to a \mathbb{G}_m -action (use the inclusion $\mathbb{G}_m = \{0\} \times \mathbb{G}_m \subset B_2$), which is fiber preserving and elliptic on every fiber \mathbb{A}^2 . The curve C then consists of the sources of that \mathbb{G}_m -action.

Using the local chart description of Q given before the proof of Proposition 5.6, we see that each of the following $n+5$ functions $f_0, f_1, g_0, \dots, g_{n+1}, h$ is a well defined regular function on Q . The first line gives their definitions on Q_0 and the second line gives their definitions on Q_1 .

$$\begin{array}{ccc|ccc} f_i: Q & \rightarrow & \mathbb{C} & g_i: Q & \rightarrow & \mathbb{C} & h: Q & \rightarrow & \mathbb{C} \\ (t, (u, v)) & \mapsto & t^{1-i}v & (t, (u, v)) & \mapsto & t^{n+2-i}u + t^{n+1-i}v^n & (t, (u, v)) & \mapsto & u \\ (t, (u, v)) & \mapsto & t^i v & (t, (u, v)) & \mapsto & t^i u & (t, (u, v)) & \mapsto & t^{n+2}u - t^{n+1}v^n. \end{array}$$

Inspired by algebraic relations between these functions, we define a variety as follows:

Definition 5.8. Let $\hat{P} \hookrightarrow \mathbb{A}^{n+5}$ be the reduced affine variety which is given by the ideal

$$I := \left\langle \begin{array}{l} (Y_i Y_j - Y_k Y_l)_{i+j=k+l}, \\ (X_0 Y_{i+1} - X_1 Y_i)_{i=0, \dots, n}, \\ (Z Y_i + X_1^n Y_{i+1} - Y_{i+1} Y_{n+1})_{i=0, \dots, n}, \\ Z X_0 + X_1^{n+1} - X_1 Y_{n+1} \end{array} \right\rangle \subset \mathbb{C}[X_0, X_1, Y_0, \dots, Y_{n+1}, Z].$$

As a preparation for Proposition 5.11, where we study the affinization morphism of Q , we make the following observation.

Lemma 5.9. Let S_0 and S_1 be the closed subsets of \hat{P} which are given respectively by

$$S_0: X_1 = Z = 0 \quad \text{and} \quad S_1: X_0 = Y_0 = 0.$$

Then $S_0 \cap S_1 = \{0\}$.

Proof. Suppose that $p = (a_0, a_1, b_0, \dots, b_{n+1}, c) \in S_0 \cap S_1$. Then $a_0 = a_1 = b_0 = c = 0$, and using the relations given by the ideal I , we get $b_1 b_{n+1} = 0$. If $b_{n+1} = 0$, we get $b_i = 0$ for all i by induction since $b_i^2 = b_{i-1} b_{i+1}$ for $i = n, n-1, \dots, 1$. If $b_{n+1} \neq 0$, all b_i are zero, since $b_{i+1} b_{n+1} = 0$. We get $p = 0 \in \hat{P}$ in any case. □

Remark 5.10. Using the relations given by I , one can check that $S_i \simeq \mathbb{A}^2$ for $i = 0, 1$.

Proposition 5.11. The morphism

$$\psi: Q \rightarrow \hat{P}, \quad q \mapsto (f_0, f_1, g_0, \dots, g_{n+1}, h)(q)$$

contracts the curve $C \simeq \mathbb{P}^1$ given by $u = v = 0$ (in both Q_0 and Q_1) to $0 \in \hat{P}$. The restriction $\psi|_{Q \setminus C}: Q \setminus C \xrightarrow{\sim} \hat{P} \setminus \{0\}$ is an isomorphism.

Proof. We check that ψ induces isomorphisms

$$\hat{P} \setminus S_i \xrightarrow{\sim} Q_i \setminus (Q_i \cap C), \quad i = 0, 1.$$

The restriction of ψ to Q_1 is given by

$$\psi|_{Q_1}: (t, (u, v)) \mapsto (v, tv, u, tu, \dots, t^{n+1}u, t^{n+2}u - t^{n+1}v^n).$$

Note that $\psi(Q \setminus Q_1)$ is the image of $t = 0$ in the chart Q_0 . It follows that $\psi(Q \setminus Q_1) \subset S_1$ with S_1 as in Lemma 5.9, and we may define an inverse map locally:

$$\begin{aligned} \hat{P} \setminus S_1 &\xrightarrow{\sim} Q_1 \setminus (Q_1 \cap C) \\ (a_0, a_1, b_0, \dots, b_{n+1}, c) &\mapsto \begin{cases} (a_1/a_0, b_0, a_0) & \text{if } a_0 \neq 0 \\ (b_1/b_0, b_0, a_0) & \text{if } b_0 \neq 0. \end{cases} \end{aligned}$$

Analogously, we define an inverse of

$$\psi|_{Q_0}: (t, (u, v)) \mapsto (tv, v, t^{n+2}u + t^{n+1}v^n, t^{n+1}u + t^n v^n, \dots, tu + v^n, u)$$

as follows:

$$\begin{aligned} \hat{P} \setminus S_0 &\xrightarrow{\sim} Q_0 \setminus (Q_0 \cap C) \\ (a_0, a_1, b_0, \dots, b_{n+1}, c) &\mapsto \begin{cases} (a_0/a_1, c, a_1) & \text{if } a_1 \neq 0 \\ ((b_{n+1} - a_1^n)/c, c, a_1) & \text{if } c \neq 0. \end{cases} \end{aligned}$$

One can check, using the relations given by the ideal I in Definition 5.8, that these morphisms indeed are isomorphisms. \square

Remark 5.12. The variety \hat{P} is an SL_2 -embedding since Q is, and we only contracted a curve of fixed-points in $Q \setminus \mathrm{SL}_2$. It is also clear that \hat{P} is three dimensional with the origin as its only singular point.

Remark 5.13. Restricting ψ to SL_2 , we get

$$\begin{aligned} \psi|_{\mathrm{SL}_2}: \mathrm{SL}_2 &\rightarrow \hat{P} \\ \begin{pmatrix} x & u \\ y & v \end{pmatrix} &\mapsto (y, x, vy^{n+1}, vxy^n, \dots, vx^{n+1}, ux^{n+1}), \end{aligned}$$

so in particular we have $\psi^*(f_0) = y$ and $\psi^*(f_1) = x$.

We start our preparations for the proof of Proposition 5.16.

Definition 5.14. Let us define the bidegree of nonzero monomials in $\mathbb{C}[v, t, u]$ as $\mathrm{bideg}(ct^i v^j u^k) = (k, j) \in \mathbb{N}^2$, $\mathrm{bideg}(0) = (-\infty, -\infty)$, and then we extend this to a function $\mathrm{bideg}: \mathbb{C}[v, t, u] \rightarrow \mathbb{N}^2 \cup \{(-\infty, -\infty)\}$ by taking the maximal bidegree of the terms, with respect to the lexicographical order. For example $\mathrm{bideg}(t^7 u^5 + 2v^3 u^4 + 3vu^5) = (5, 1)$.

Lemma 5.15. Let $F \in \mathbb{C}[t, v, u]$ be a nonzero polynomial with bideg-leading term $ct^i v^j u^k$, where $i \leq j + kn + 2k$. Then $F = \tilde{F} + L$ for some

$$\tilde{F} \in \mathbb{C}[v, vt, u, ut, \dots, ut^{n+1}, ut^{n+2} - v^n t^{n+1}] \quad \text{and} \quad L \in \mathbb{C}[t, v, u]$$

with $\mathrm{bideg}(L) < \mathrm{bideg}(F)$.

Proof. If $i \leq j$, we take $\tilde{F} = c(tv)^i v^{j-i} u^k$. If $i > j$, we find integers q, r so that $i - j = (n + 2)q + r$ with $0 \leq r \leq n + 1$ and $0 \leq q \leq k - 1$, and then we take $\tilde{F} = c(tv)^j ut^r (ut^{n+2} - v^n t^{n+1})^q u^{k-q-1}$. In both cases $\mathrm{bideg}(F - \tilde{F}) < \mathrm{bideg}(F)$. \square

Proposition 5.16. The $n + 5$ functions $f_0, f_1, g_0, \dots, g_{n+1}, h$ generate $\mathcal{O}(Q)$ as a \mathbb{C} -algebra, and $\psi^*: \mathcal{O}(\hat{P}) \xrightarrow{\sim} \mathcal{O}(Q)$ is an isomorphism.

Proof. Since Q is normal and C of codimension 2 we get a morphism

$$\mathcal{O}(\hat{P}) \xrightarrow{\psi^*} \mathcal{O}(Q) \simeq \mathcal{O}(Q \setminus C) \simeq \mathcal{O}(\hat{P} \setminus \{0\}).$$

It is clearly injective, and the surjectivity follows from $\mathcal{O}(Q) = \mathbb{C}[f_0, f_1, g_0, \dots, g_{n+1}, h]$, a fact that we now prove:

The regular functions on Q are the elements of $\mathbb{C}(Q)$ which are defined everywhere on Q_0 and Q_1 , i.e. they can be seen as the polynomial functions on $\mathcal{O}(Q_1) = \mathbb{C}[u, v, t]$ which remain polynomial as functions on Q_0 after the coordinate change induced by the transition function $V_0 \rightarrow V_1$ (c.f. Definition 5.5).

Let $F \in \mathcal{O}(V_1) = \mathbb{C}[u, v, t] \subset \mathbb{C}(Q)$ be a nonzero regular function on Q with bideg-leading term $ct^i v^j u^k$. After the coordinate change, the bideg-leading term of F becomes

$$ct^{-i}(tv)^j(ut^{n+2})^k.$$

It follows in particular, since F is regular on Q , that $t^{-i+j+kn+2k}$ is a polynomial, so $i \leq j + nk + 2k$. Now we use

Lemma 5.15 in order to write $F = \tilde{F} + L \in \mathcal{O}(V_1)$ with $\tilde{F} \in \mathbb{C}[v, vt, u, ut, \dots, ut^{n+1}, ut^{n+2} - v^n t^{n+1}]$ and $\text{bideg}(L) < \text{bideg}(F)$. Repeating this procedure a finite number of times, we finally arrive at $L = 0$ and the claim of the proposition follows. \square

Finally, we announce the result which settles part (1) of Theorem 4

Proposition 5.17. *We have*

$$\text{gr}_D(\mathcal{O}(\hat{P})) \simeq \mathbb{C}[x, y] \oplus \bigoplus_{\nu \geq 1} \langle x, y \rangle^{(n+2)\nu} t^\nu.$$

Proof. Using Remark 5.7, we see that the \mathbb{G}_a -action on \hat{P} corresponds to the derivation $D: \mathcal{O}(\hat{P}) \rightarrow \mathcal{O}(\hat{P})$ given by $D(f_i) = 0$, $D(g_i) = f_1^i f_0^{n+2-i}$, $D(h) = f_1^{n+2}$. It follows that $D^\nu(\mathcal{O}(\hat{P})_{\leq \nu}) = \langle f_0, f_1 \rangle^{(n+2)\nu} \subset \mathcal{O}(\hat{P})$ for each $\nu \geq 0$. Hence

$$\text{gr}_D(\mathcal{O}(\hat{P})) = \mathbb{C}[f_0, f_1] \oplus \bigoplus_{\nu=1}^{\infty} \langle f_0, f_1 \rangle^{(n+2)\nu} t^\nu.$$

This finishes the proof since $\psi^*(f_0) = y$ and $\psi^*(f_1) = x$ (as in Remark 5.13). \square

Proposition 5.18. *The variety \hat{P} is normal.*

Proof. Suppose that $f \in \mathbb{C}(Q)$ is integral over $\mathcal{O}(Q)$. Then it is in particular integral over $\mathcal{O}_{Q,q}$ for each $q \in Q$, and since Q is normal, it follows that $f \in \bigcap_{q \in Q} \mathcal{O}_{Q,q} = \mathcal{O}(Q)$. Thus $\mathcal{O}(Q) \simeq \mathcal{O}(\hat{P})$ is integrally closed and since \hat{P} is an affine variety, we are finished. \square

Proof of Theorem 3. Let $B_n := \mathcal{O}(\hat{P}_n) \subset \mathcal{O}(\text{SL}_2)$ and suppose that the affine SL_2 -extension is given by $\hat{R} = \text{Spec}(C)$. By Lemma 4.5, we can choose an n such that $\langle x, y \rangle^{n+2} \subset \mathfrak{m}_1(\hat{R})$. Then we have $(B_n)_{\leq 1} \subset C_{\leq 1}$ and thus $B_n \subset C$, since B_n is, as a $\mathbb{C}[x, y]$ -algebra, generated by $(B_n)_{\leq 1}$. \square

6. THE SECOND FAMILY OF SL_2 -EXTENSIONS

In this section we prove part (2) of Theorem 4 by constructing a family $\hat{P}(p, q)$ of affine extensions of SL_2 , depending on two relatively prime natural numbers $p, q \in \mathbb{N}_{>0}$, such that

$$\mathfrak{m}_\nu(\hat{P}(p, q)) = \bigoplus_{p\alpha + q\beta \geq (p+q)\nu} \mathbb{C}x^\alpha y^\beta.$$

The numbers p and q are fixed throughout this entire section, and we will write \hat{P} rather than $\hat{P}(p, q)$ and similarly for other objects that are introduced.

The key observation in order to construct the SL_2 -extension \hat{P} , is that we can equip SL_2 with a \mathbb{G}_m -action and then realize SL_2 as a \mathbb{G}_m -fibration over the quotient variety. The \mathbb{G}_m -action that we will use is defined as follows for $\lambda \in \mathbb{G}_m$.

$$\begin{pmatrix} x & u \\ y & v \end{pmatrix} \lambda := \begin{pmatrix} \lambda^p x & \lambda^{-q} u \\ \lambda^q y & \lambda^{-p} v \end{pmatrix}.$$

Proposition 6.1. *The quotient morphism with respect to this \mathbb{G}_m -action is given by*

$$\begin{aligned} \psi: \mathrm{SL}_2 &\rightarrow Y \simeq \{(a, b, c) \in \mathbb{A}^3 \mid ac = b^q(b-1)^p\}, \\ \begin{pmatrix} x & u \\ y & v \end{pmatrix} &\mapsto (x^q u^p, xv, y^p v^q), \end{aligned}$$

It is a \mathbb{G}_m -fibration which is a principal \mathbb{G}_m -bundle above the regular part of Y . Indeed Y has at most two singular points and it is smooth if and only if $p = q = 1$.

Proof. Since $\gcd(p, q) = 1$, we find $m, n \in \mathbb{Z}$ with $mq - np = 1$, and we find \mathbb{G}_m -equivariant trivializations given respectively by

$$\begin{aligned} Y_a \times \mathbb{G}_m &\rightarrow (\mathrm{SL}_2)_{xu} \\ ((a, b, c), \lambda) &\mapsto \begin{pmatrix} a^m \lambda^p & a^{-n} \lambda^{-q} \\ (b-1)a^n \lambda^q & ba^{-m} \lambda^{-p} \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} Y_c \times \mathbb{G}_m &\rightarrow (\mathrm{SL}_2)_{yv} \\ ((a, b, c), \lambda) &\mapsto \begin{pmatrix} bc^{-m} \lambda^p & (b-1)c^n \lambda^{-q} \\ c^{-n} \lambda^q & c^m \lambda^{-p} \end{pmatrix}. \end{aligned}$$

The variety Y is smooth at all points except possibly $(0, 0, 0)$, which is singular if and only if $q > 1$, and $(0, 1, 0)$, which is singular if and only if $p > 1$. □

Remark 6.2. The transition function from the first chart to the second in the above proof is given by $(a, b, c, \lambda) \mapsto (a, b, c, (b-1)^m b^n \lambda)$, i.e. multiplication by $(b-1)^m b^n \in \mathbb{G}_m$.

Remark 6.3. The \mathbb{G}_m -action on SL_2 has possibly nontrivial stabilizers only along two orbits, namely $\psi^{-1}(0, 1, 0)$ and $\psi^{-1}(0, 0, 0)$ with stabilizers the groups C_p and C_q of p -th and q -th roots of unity respectively.

Definition 6.4. We define

$$\hat{P} := \mathrm{SL}_2 \times^{\mathbb{G}_m} \mathbb{A}^1$$

This definition is analogous to Definition 5.5, in the sense that \hat{P} is the orbit space with respect to the action

$$\mathbb{G}_m \times (\mathrm{SL}_2 \times \mathbb{A}^1) \rightarrow \mathrm{SL}_2 \times \mathbb{A}^1, \quad (\lambda, (x, y)) \mapsto (x\lambda^{-1}, \lambda y).$$

A main difference is that $\hat{P} = (\mathrm{SL}_2 \times \mathbb{A}^1) // \mathbb{G}_m$ is already an affine variety in this situation, \mathbb{G}_m being reductive. We also get normality of \hat{P} for free, since $\mathrm{SL}_2 \times \mathbb{A}^1$ is normal.

Remark 6.5. Intuitively \hat{P} is again obtained from SL_2 by replacing the fiber \mathbb{G}_m in the fibration in Proposition 6.1 by $\mathbb{A}^1 \supset \mathbb{G}_m$ – though the replacement process itself is not as obvious as in the case of a principal bundle. In any case $\mathcal{O}(\hat{P}) \subset \mathcal{O}(\mathrm{SL}_2)$ consists of those functions which are defined as $\lambda \in \mathbb{G}_m$ tends to 0.

It is a straightforward verification to check that

$$(A * s)\lambda = (A\lambda) * (\lambda^{-(p+q)} s)$$

holds, for the standard right \mathbb{G}_a -action on SL_2 which is given by

$$\mathrm{SL}_2 \times \mathbb{G}_a \rightarrow \mathrm{SL}_2, \quad (z, s) \mapsto z * s := z \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}.$$

This is well known to be equivalent with the fact that the locally nilpotent derivation D on $\mathcal{O}(\mathrm{SL}_2)$ which corresponds to the \mathbb{G}_a -action is homogeneous of degree $p + q$ with respect to the grading which corresponds to the \mathbb{G}_m -action. This will be used in the proof of the following proposition which will settle part (2) of Theorem 4.

Proposition 6.6. *The \mathbb{G}_a -action on SL_2 extends to \hat{P} and for $B := \mathcal{O}(\hat{P})$ we have*

$$\mathfrak{m}_\nu(\hat{P}) = \bigoplus_{p\alpha + q\beta \geq (p+q)\nu} \mathbb{C}x^\alpha y^\beta.$$

Proof. The \mathbb{G}_m -action on SL_2 corresponds to a \mathbb{Z} -grading $\mathcal{O}(\mathrm{SL}_2) = \bigoplus_{\mu=-\infty}^{\infty} \mathcal{O}(\mathrm{SL}_2)(\mu)$, with respect to which the locally nilpotent derivation $D: \mathcal{O}(\mathrm{SL}_2) \rightarrow \mathcal{O}(\mathrm{SL}_2)$ is homogeneous of degree $p + q$. Since x, y, u, v are of degree $p, q, -q, -p$ respectively, we have

$$\mathcal{O}(\mathrm{SL}_2)(\mu) = \sum_{p(i-\ell) + q(j-k) = \mu} \mathbb{C}x^i y^j u^k v^\ell$$

and as in Example 4.4 we have

$$\mathcal{O}(\mathrm{SL}_2)_{\leq \nu} = \bigoplus_{k+\ell=\nu} \mathbb{C}[x, y]u^k v^\ell$$

for the D -filtration of $\mathcal{O}(\mathrm{SL}_2)$.

It follows from Remark 6.5 and the definition of $\mathcal{O}(\mathrm{SL}_2)(\mu)$ that $B = \bigoplus_{\mu=0}^{\infty} \mathcal{O}(\mathrm{SL}_2)(\mu)$ is the non-negative part of the \mathbb{Z} -graded algebra $\mathcal{O}(\mathrm{SL}_2)$. In particular, the \mathbb{G}_a -action on SL_2 extends to an action on \hat{P} , since B is D -invariant, D being homogeneous of degree $p + q \geq 2$.

Let us now determine $\mathrm{gr}_D(B)$ using the \mathbb{G}_m -decomposition of $\mathcal{O}(\mathrm{SL}_2)$. Since the locally nilpotent derivation $D: \mathcal{O}(\mathrm{SL}_2) \rightarrow \mathcal{O}(\mathrm{SL}_2)$ is homogeneous, the \mathbb{G}_m -grading descends to the associated graded algebra

$$\mathrm{gr}_D(\mathcal{O}(\mathrm{SL}_2)) = \mathbb{C}[x, y] \oplus \bigoplus_{\nu=1}^{\infty} \langle x, y \rangle^\nu t^\nu,$$

where the \mathbb{G}_m -grading on $\mathbb{C}[x, y]$ satisfies $\deg(x) = p, \deg(y) = q$ and associates the degree $-(p + q)$ to the variable t (though $t \notin \mathrm{gr}_D(\mathcal{O}(\mathrm{SL}_2))$). Indeed $xt = \mathrm{gr}(u)$ and $yt = \mathrm{gr}(v)$.

It follows that

$$\mathrm{gr}_D(B) = \mathrm{gr}_D(\mathcal{O}(\mathrm{SL}_2))_{\geq 0} \hookrightarrow \mathrm{gr}_D(\mathcal{O}(\mathrm{SL}_2)),$$

where the subscript refers to the \mathbb{G}_m -grading. In other words,

$$\mathrm{gr}_D(B) = \mathbb{C}[x, y] \oplus \bigoplus_{\nu=1}^{\infty} \mathfrak{m}_\nu t^\nu$$

with

$$\mathfrak{m}_\nu = (\langle x, y \rangle^\nu)_{\geq \nu(p+q)} = \bigoplus_{\alpha p + \beta q \geq \nu(p+q)} \mathbb{C}x^\alpha y^\beta.$$

□

7. SMALL FIXED POINT SETS

It follows from Proposition 4.8 that the exceptional fiber $E = \hat{P}_n \setminus \mathrm{SL}_2$ consists of \mathbb{G}_a -fixed points for the SL_2 -extensions in the family that was constructed in section 5. For the extensions $\hat{P} = \mathrm{Spec}(B)$ from section 6 it goes the same. This follows from the fact that $D(f) \in \bigoplus_{\mu=p+q}^{\infty} \mathcal{O}(\mathrm{SL}_2)(\mu)$ for all $f \in B$. But since $p+q > 0$, this implies that the exceptional fiber $E = \hat{P} \setminus \mathrm{SL}_2$ consists of fixed points of the \mathbb{G}_a -action, as all functions of positive \mathbb{G}_m -degree vanish as $\lambda \in \mathbb{G}_m$ tends to 0.

In this section, we construct some extensions with empty fixed point set and with one dimensional fixed point set, taking the SL_2 -extension $\hat{P}(1, 1)$ as starting point.

Proposition 7.1. *Assume that the exceptional fiber $E \hookrightarrow Y = \mathrm{Spec}(B)$ of an affine extension $Y \rightarrow \mathbb{A}^2$ of $P \rightarrow \mathbb{A}_*^2$ is (the support of) a Cartier divisor and coincides with the fixed point set $E = Y^{\mathbb{G}_a}$. Let $C \hookrightarrow E$ be a closed subvariety. Denote $\mathrm{Bl}_C(Y) \rightarrow Y$ the blowup of Y with center $C \hookrightarrow E$ and $\tilde{E} \hookrightarrow \mathrm{Bl}_C(Y)$ the strict transform of E . Then*

$$Y_1 := \mathrm{Bl}_C(Y) \setminus \tilde{E}$$

is an affine extension.

Proof. It is clear that Y_1 inherits a \mathbb{G}_a -action, since the center of the blowup is fixed by the \mathbb{G}_a -action on Y . It remains to show Y_1 is affine. It is enough to check that the morphism $Y_1 \rightarrow Y$ is affine. After passing to a cover of affine open subsets of $\mathrm{Spec}(B)$, we may assume that $E \hookrightarrow Y$ is given by one function: $I(E) = (f) \subset B = \mathcal{O}(Y)$. But then, if $I(C) = (g_1, \dots, g_s) \ni f$ we have

$$Y_1 = \mathrm{Spec}(B[\frac{g_1}{f}, \dots, \frac{g_s}{f}]),$$

see [KaZa99, Prop. 1.1]. □

Now we start with $Y = \hat{P}_0 \supset \mathrm{SL}_2$ and take $Y_1 := \mathrm{Bl}_a(Y) \setminus \tilde{E}$ with the exceptional fiber $E \hookrightarrow \hat{P}_0$ and some point $a \in E$. Using the realization of Y as locally trivial bundle over \mathbb{P}^1 , we see that we may think of $a \in Y$ as the origin in

$$a = (0, 0, 0) \in \mathbb{A}^3 = \mathrm{Spec}(\mathbb{C}[x, y, z]) =: U$$

where $(x, y, z) \mapsto [1 : z]$ is the bundle projection, while $\mathbb{A}^1 \times \mathbb{A}_*^1 \times \mathbb{A}^1$ is $\mathbb{G}_a \rtimes_{\sigma} \mathbb{G}_m \times \mathbb{A}^1$. Furthermore the \mathbb{G}_a -action corresponds to

$$D: \mathbb{C}[x, y, z] \rightarrow \mathbb{C}[x, y, z], x \mapsto y^2, y \mapsto 0, z \mapsto 0$$

and $E \cap U = \mathbb{A}^1 \times 0 \times \mathbb{A}^1$. Then above U , in the blowup, we have

$$U_1 := \mathrm{Spec}(\mathbb{C}[x/y, y, z/y]),$$

with $D(x/y) = y$, and $D(y) = 0 = D(z/y)$. If we take $\xi = x/y, \eta = y, \zeta = z/y$, we see that \mathbb{G}_a acts linearly on $U_1 = \mathbb{A}^3 = \mathrm{Spec}(\mathbb{C}[\xi, \eta, \zeta])$, namely $D = \eta \frac{\partial}{\partial \xi}$.

Note that

$$Y_1 \supset U_1 \supset E_1 = \mathbb{A}^1 \times 0 \times \mathbb{A}^1.$$

Now let us apply the recipe of Proposition 7.1 with some subvariety $C \hookrightarrow E_1$. We obtain an affine extension Y_2 . We discuss several choices of C :

- (1) If $C = \{(0, 0, 0)\}$, the exceptional fiber is naturally isomorphic to

$$E_2 \simeq \mathbb{P}(\mathbb{A}^3) \setminus \mathbb{P}(\mathbb{A}^1 \times 0 \times \mathbb{A}^1)$$

with the restriction of the induced linear \mathbb{G}_a -action on $\mathbb{P}(\mathbb{A}^3) \simeq \mathbb{P}(T_0(\mathbb{A}^3))$. Hence the \mathbb{G}_a -action on Y_2 is free.

(2) Now let $C \hookrightarrow E_1 = \mathbb{A}^1 \times 0 \times \mathbb{A}^1$ be a smooth curve. For the fiber F_b over a point $b \in C$ there is a natural isomorphism

$$F_b \simeq \mathbb{P}(\mathbb{A}^3/T_b(C)) \setminus \{(\mathbb{A}^1 \times 0 \times \mathbb{A}^1)/T_b(C)\}.$$

We distinguish two cases:

- (1) If $T_b(C) = \mathbb{C}(1, 0, 0)$, the \mathbb{G}_a -action on F_b is trivial.
- (2) If $T_b(C) = \mathbb{C}(\alpha, 0, \beta)$ with $\beta \neq 0$, the \mathbb{G}_a -action on F_b is free. So if C is not a line parallel to $\mathbb{C}(1, 0, 0)$, the fixed point set has at most dimension one.

REFERENCES

- [DuFi11] ADRIEN DUBOULOZ AND DAVID R. FINSTON: *On exotic affine 3-spheres*. J. Algebraic Geom. **23** (2014), 445–469
- [Nag59] M. NAGATA: *Lectures on the Fourteenth Problem of Hilbert* Lecture Notes, Tata Institute, Bombay, 31, (1959)
- [Fie94] K-H. FIESELER: *On complex affine surfaces with \mathbb{C}^+ -action*. Comment. Math. Helv. **69** (1994), no. 1, 5–27. MR 1259603 (95b:14027)
- [KaZa99] S. KALIMAN AND M. ZAIDENBERG: *Affine modifications and affine hypersurfaces with a very transitive automorphism group*. Transformation groups, Vol. 4, No. 1, 1999, pp. 53–95.
- [Hor64] G. HORROCKS: *Vector bundles on the punctured spectrum of a local ring*. Proc. London Math. Soc. (3) **14** (1964), 689–713. MR0169877 (30#120).
- [TLam06] T.Y. LAM: *Serre’s problem on projective modules*. Springer monographs in mathematics, Springer, 2006.

ISAC HEDÉN, MATHEMATISCHES INSTITUT, UNIVERSITÄT BASEL, RHEINSPRUNG 21, 4051 BASEL, SWITZERLAND

E-mail address: Isac.Heden@unibas.ch